# Spectral Properties of Spiral Waves in the Karma Model

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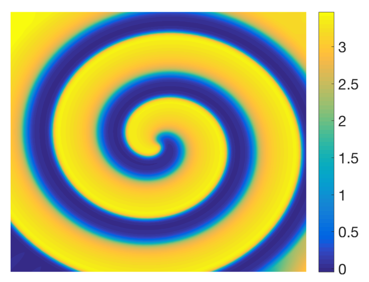
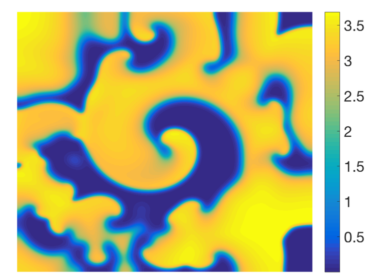
## Spiral Waves in Cardiac Arrhythmias

- **Cardiac Arrhythmia** refers to abnormal electrical activity in the heart.



**Ventricular Tachycardia:** (VT) Reentrant spiral waves create self-sustained oscillations.



**Ventricular Fibrillation:** (VF) Spiral wave breakup leads to unorganized self-sustained electrical activity.

- VF may lead to sudden cardiac death, which is responsible for $> 350{,}000$ deaths/year.
- **Alternans** is a marker for sudden cardiac death.

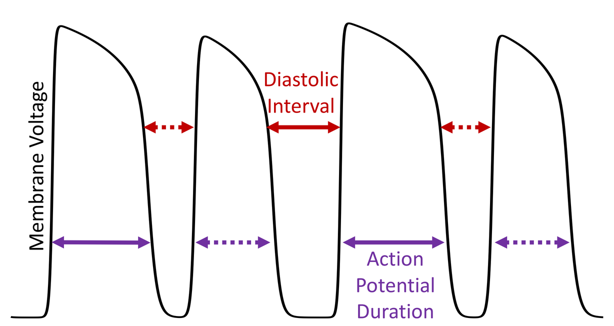



Figure: Cartoon of alternans. Membrane voltage exhibits periodic variation in length of action potential duration and diastolic interval.

## Goals

The aim of this research is to investigate what spectral properties can tell us about the stability of spiral waves in cardiac arrhythmias, in particular alternans instability.

- Understand and illustrate properties of spiral spectra.
- Relate spectral properties to alternans instability observed in spirals.

## The Karma Model

A simple reaction-diffusion cardiac model that exhibits alternans.

$$\begin{cases} E_t = \gamma\Delta E + \dfrac{1}{\tau_E}\left(-E + [E^* - n^M][1-\tanh(E-E_h)]\dfrac{E^2}{2}\right) \\ n_t = \delta\Delta n + \dfrac{1}{\tau_n}\left(\dfrac{1}{1-e^{-Re}}\theta(E-E_n) - n\right) \end{cases}$$

- $E(x,t)$ describes membrane voltage and $n(x,t)$ provides slower dynamics.
- $E_n$, $E_h$, $E^*$, $\delta$, $\tau_E$, $\tau_n \in \mathbb{R}$ control excitable threshold and fast/slow timescale.
- $Re \in \mathbb{R}$ controls slope of restitution curve.

Written as a system in polar coordinates, the model is

$$U_t = D\Delta_{r,\phi}U + F(U), \quad U = \begin{pmatrix} E \\ n \end{pmatrix}(r,\phi), \quad D = \begin{pmatrix} \gamma & 0 \\ 0 & \delta \end{pmatrix}.$$

**Rigidly rotating spiral waves**, $U^*(r,\psi)$, are stationary solutions in a rotating polar frame, $(r,\phi) \to (r,\psi) = (r, \phi - \omega t)$

$$0 = D\Delta_{r,\psi}U^* + \omega U^*_\psi + F(U^*).$$

Spirals tend to 1D periodic **asymptotic wave trains**, $U^\infty$, as $r \to \infty$

$$U^*(r,\psi) \to U^\infty(\kappa r + \psi) = U^\infty(\xi), \quad U^\infty(\xi) = U^\infty(\xi + 2\pi).$$

Wave trains are stationary solutions of

$$U_t = \kappa^2 D U_{\xi\xi} + \omega U_\xi + F(U).$$

## Acknowledgements

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## Types of Spectra

**Temporal Eigenvalues**, $\lambda$, describe temporal growth of perturbations

$$\mathcal{L}U = D\Delta_{r,\psi}U + \omega U_\psi + F'(U^*)U = \lambda U.$$

**Spatial Eigenvalues**, $\nu$, describe the spatial growth of eigenfunctions.

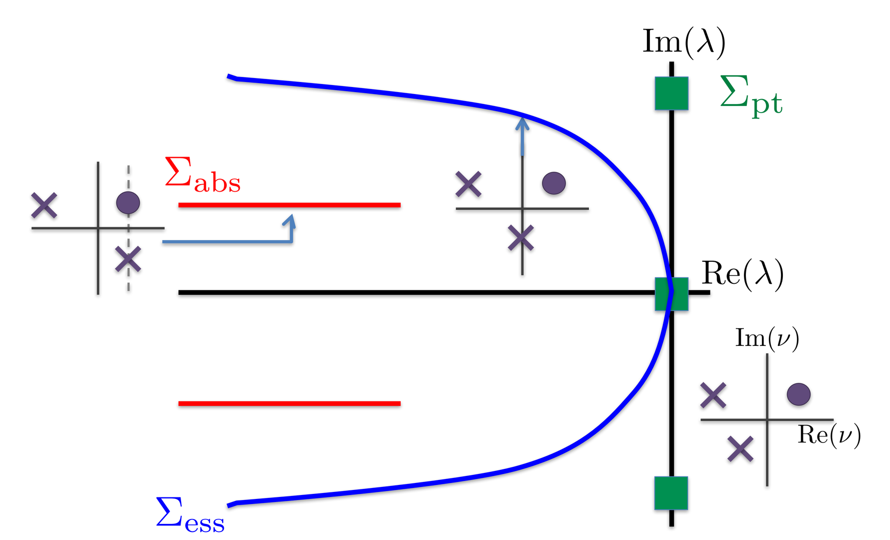


Figure: Cartoon of essential, absolute, and point spectra. Inserts show distribution of spatial eigenvalues.

**On the plane:**

- Point spectrum, $\Sigma_{\text{pt}}$
- Essential spectrum, $\Sigma_{\text{ess}}$
  - $(\lambda - \mathcal{L})$ is not Fredholm
  - $\nu \in i\mathbb{R}$

**On bounded domain:**

- Point spectrum, $\Sigma_{\text{pt}}$
- Absolute spectrum, $\Sigma_{\text{abs}}$
  - Limit of discrete spectrum as domain $\to \infty$
  - No longer separate stable/unstable spatial eigenvalues

Dispersion relations of spiral, $\lambda_*(\nu_*)$, and wave train, $\lambda_\infty(\nu_\infty)$, are related via

$$\lambda_*(\nu_*) = \lambda_\infty(\nu_\infty) - \omega\nu_\infty + i\omega\ell, \quad \ell \in \mathbb{Z}, \quad \nu_* = \kappa\nu_\infty.$$

## Alternans is Preceded by Meandering

- Known that Hopf bifurcation leads to meandering.

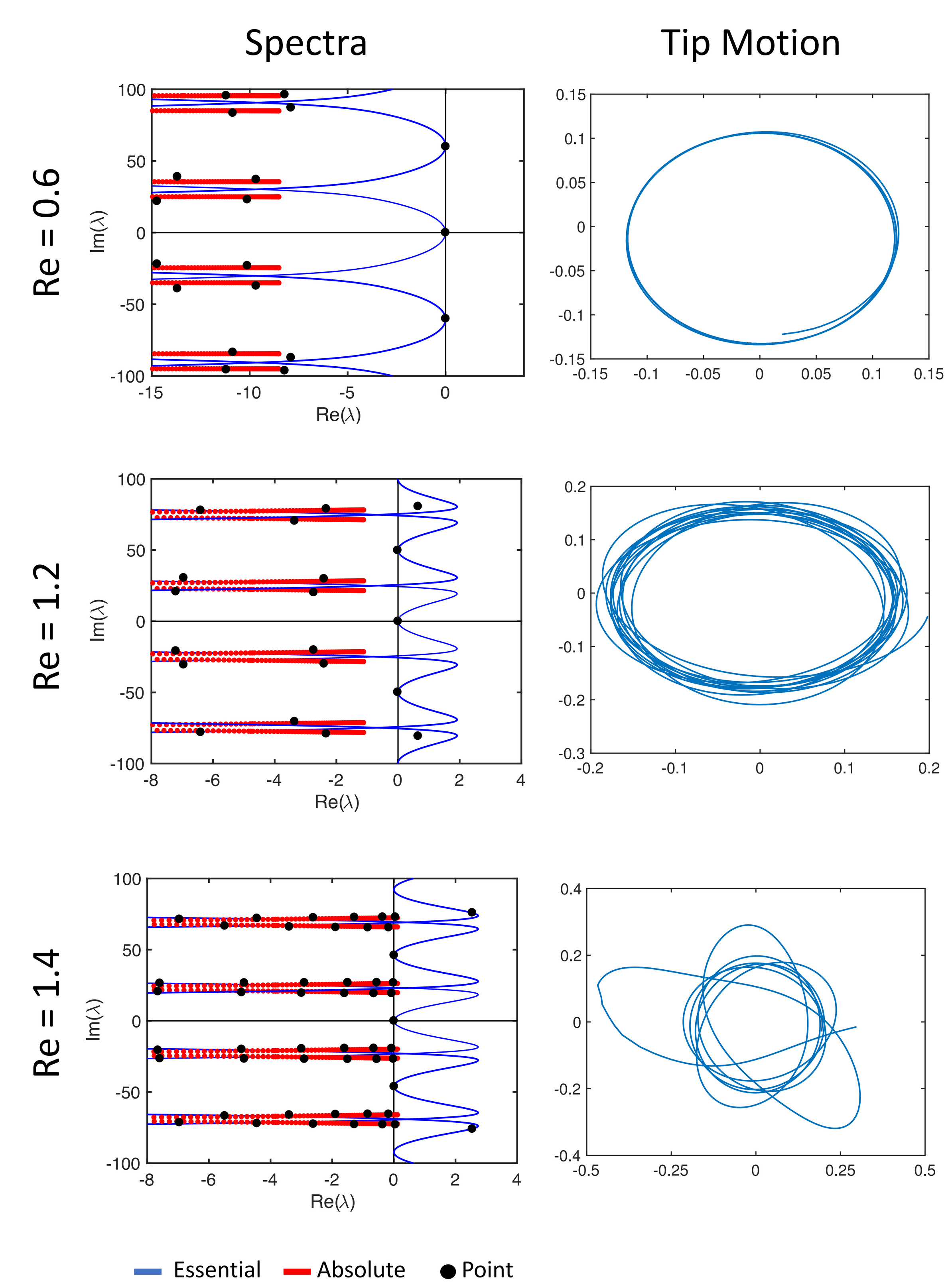


Figure: Spectra and tip motion of spirals in the Karma Model. Point spectrum calculated from spiral on 5 cm bounded disk, absolute and essential spectrum from wave trains. Additional parameters: $E_h = 3$, $E_n = 1$, $E^* = 1.5414$, $\tau_E = 0.0025$, $\tau_n = 0.25$, $\gamma = 1.1$, $\delta = 0.1$.

## Unstable Eigenfunctions

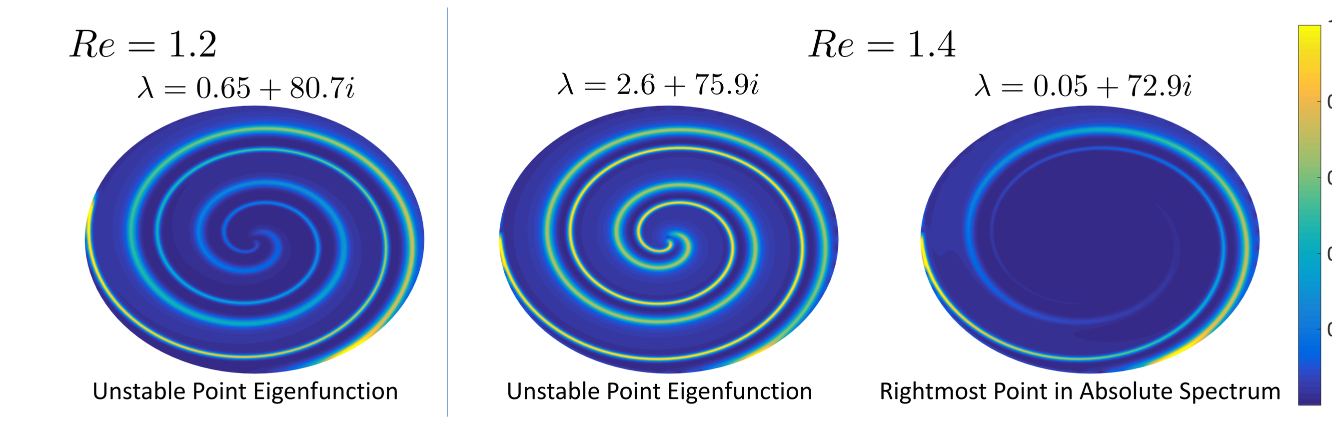


Figure: Absolute value of $E$-component of unstable eigenfunctions for radius 5 cm spiral.

- Growth toward boundary in $Re = 1.2$ unstable eigenfunction.
- Unstable point eigenfunction in $Re = 1.4$ interacts with essential spectrum and has global behavior.
- Eigenfunctions in absolute spectrum are localized away from spiral core.

## Point and Absolute Spectrum Leads to Planar Alternans

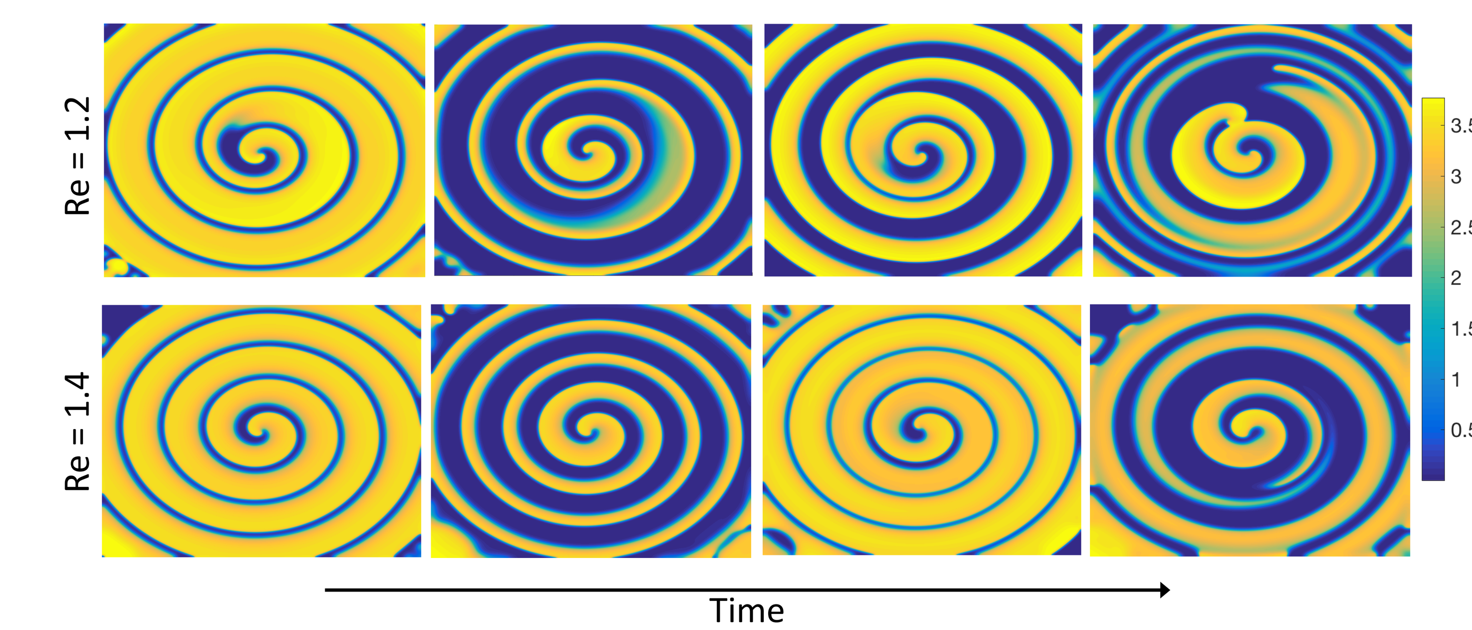


Figure: Spiral breakup due to alternans in the Karma Model on a 16 cm x 16 cm square with Neumann boundary conditions. Top row shows $Re = 1.2$, bottom is $Re = 1.4$. Color-bar indicates the membrane voltage. Solutions evolved on fourth-order finite-difference spatial grid using Crank-Nicholson and Adams-Bashforth IMEX scheme.

## Conclusions

- Spiral break up occurs as bands collide and form conduction blocks.
- Form of unstable eigenfunctions shows expansion/compression of spiral bands associated with alternans.
- Alternans instability likely caused by unstable eigenfunctions in the point spectrum.

## Future Work

- Use 1D eigenfunctions to learn about and predict the shape of instabilities.
- Evaluate contributions to spiral break up from point and absolute spectrum.
- Determine if Hopf bifurcations are super or subcritical.
- Analyze case when one or more variables are diffusionless.
- Investigate alternans instability in other cardiac models.

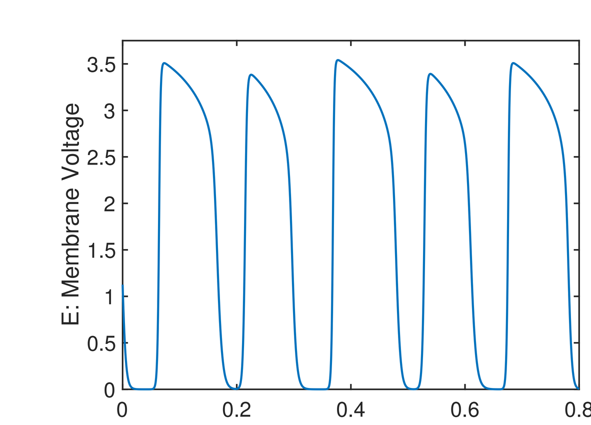


Figure: Alternans in 1D Karma Model.

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